Morita Equivalence and Interpolation of The Dirac-Born-infeld Theory on the Non-Commutative Torus

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[1]

Abstract

In the noncommutative Dirac-Born-Infeld action with Chern-Simons term, an interpolation field Φ is used in both DBI action and Chern-Simons term. The Morita equivalence is discussed in both the lagrangian and the Hamiltonian formalisms, which is more transparent in this treatment.

In recent two years, the application of noncommutative geometry in string/M theory has got great development. This started with a paper for describing M-theory compactifying on a noncommutative two-torus [1]. Following it there appeared a lot of papers, some emphasizing matrix model[2], others emphasizing D-brane [3] (a more complete list can be found from [4]). In these researches, a kind of new symmetry called Morita equivalance has been studied extensively [5-10]. The correlation with the T-duality of type II string is also elaborated [8,9,11].

In reference [4], Seiberg and Witten derived noncommutative Yang-Mills theory from the quantization of open string, ending on D-brane in the presence of a NS-NS B-field. They argued that whether the commutativity or noncommutativity of Yang-Mills theory depends on the choice of regularization and proved their equivalence through the DBI action. Especially they proposed that there is an interpolating theory between these two with a modulus Φ which can be considered as a magnetic background [6,11].

Using this interpolating scheme Ryang examined the Morita equivalence of DBI action [10]. He got same Morita transformation rule for the noncommutative open string parameters as reference [4] but without recourse to the low energy zero slope limit $\alpha' \to 0$. His proof of the Morita transformation invariance is not only directly in DBI Lagrangian form, but also much simpler than other methods (for example Ref. [9]). However, he didn't let the interpolation cover the whole action, which includes a Chern-Simons topological term. The C fields in this term change in a somehow complicated way under Morita transformation. Because of this reason we would like to suggest the interpolating with modulus Φ in both the DBI action and the Chern-Simons term.

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Morita equivalence is related to the T-duality of type-II string in which D-branes are compactified on a p-torus. So we need to consider the Dirac-Born-Infeld action. For the convenience of comparision we use the same notations as references [4,10]. For ordinary D-brane we have

$$\begin{aligned} S &= \int \mathrm{d}^{p+1}\sigma\mathcal{L} \quad , \quad \mathcal{L}=\mathcal{L}_{DBI}+\mathcal{L}_{wz}, & (1)\\ \mathcal{L}_{DBI} &= -\frac{1}{g_s(2\pi)^p(\alpha')^{\frac{p+1}{2}}}STr\sqrt{-det_{(p+1)}(g+2\pi\alpha'(F+B))}, & (2) \end{aligned}$$

in which g_s, g, B belong to closed string parameters, and we use the symmetric trace for non-Abelian gauge group [12].

$$\mathcal{L}_{wz} = STr(e^{2\pi\alpha'(F+B)}\sum C_{(n)}) = STrP_{(p)}(C,F+B), \qquad (3)$$

where $P_{(p)}(C,F+B)$ is a polynomial of R-R potentials and the second equality is valid under the integral of world volume of D-p branes. But we are interested in the noncommutative D-brane with modulus Φ. Thus the Lagrangian is

$$\begin{aligned} \hat{\mathcal{L}}_{DBI} &= -\frac{1}{G_s(2\pi)^p\alpha'^{\frac{p+1}{2}}}STr_\theta\sqrt{-det_{(p+1)}(\hat{G}+\mathcal{F})}, & (4)\\ \hat{\mathcal{L}}_{wz} &= STr_\theta P_{(p)}(\hat{C},\mathcal{F}), \qquad \hat{\mathcal{L}}=\hat{\mathcal{L}}_{DBI}+\hat{\mathcal{L}}_{wz}=Str_\theta L. & (5) \end{aligned}$$

in which $\mathcal{F}\equiv 2\pi\alpha'(\hat{F}+\Phi), G_s, \hat{G}$ belong to open string and $\det_{(p+1)}$ stands for the determinent of $(p+1)\times(p+1)$ matrix (including $\hat{G}_{00}, \mathcal{F}_{0i}$ as matrix elements). Up to D-6 brane the polynomials $P_{(p)}$ of R-R potentials for D_p branes are (here we omit the hat $\wedge$ until equation(14))

$$\begin{aligned} &D1 \quad 2\pi\alpha' F_{0i}C,\\ &D2 \quad 2\pi\alpha'\epsilon^{ij}F_{0i}C_j,\\ &D3 \quad 2\pi\alpha'\epsilon^{ijk}F_{0i}(\tfrac{1}{2}C_{jk}+\tfrac{1}{2}\mathcal{F}_{jk}C),\\ &D4 \quad 2\pi\alpha'\epsilon^{ijkl}F_{0i}(\tfrac{1}{3!}C_{jkl}+\tfrac{1}{2}\mathcal{F}_{jk}C_l) = 2\pi\alpha' F_{0i}({}^\star C^i_{(3)} +{}^\star \mathcal{F}^{ij}C_{(1)j}),\\ &D5 \quad 2\pi\alpha'\epsilon^{ijklm}F_{0i}(\tfrac{1}{4!}C_{jklm}+\tfrac{1}{4}\mathcal{F}_{jk}C_{lm}+\tfrac{1}{8}\mathcal{F}_{jk}\mathcal{F}_{lm}C),\\ &D6 \quad 2\pi\alpha'\epsilon^{ijklmn}F_{0i}(\tfrac{1}{5!}C_{jklmn}+\tfrac{1}{12}\mathcal{F}_{jk}C_{lmn}+\tfrac{1}{8}\mathcal{F}_{jk}\mathcal{F}_{lm}C_n). \end{aligned} \qquad (6)$$

Even $p's$ are part of IIA string while odd $p's$ are part of IIB string. In the above result we have used the same assumption $g_{0i}=B_{0i}=0$ and $\theta_{0i}=0$ as reference [4,10]. Hence we have also $G_{0i}=\Phi_{0i}=0$. We also omit the time component of R-R potentials. Their behavior under the Morita transformation can be obtained similarly, here we only consider R-R potentials with indeces in directions on a torus T^p.

The interpolating formula proposed by Seiberg and Witten is [4],

$$\frac{1}{G+2\pi\alpha'\Phi} = -\frac{\theta}{2\pi\alpha'} + \frac{1}{g+2\pi\alpha' B}. \tag{7}$$

Introduce $E = \frac{r^2(g+2\pi\alpha' B)}{\alpha'}$ and $\Theta = \frac{\theta}{2\pi r^2}$, in which $2\pi r$ is the period for the D-p brane compactified on a torus T^p [4]. Now the T-duality SO(p,p;Z) transformations are chosen as

$$E' = (aE+b)(cE+d)^{-1}, \tag{8}$$

and

$$\Theta' = (c+d\Theta)(a+b\Theta)^{-1}, \tag{9}$$

where

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SO(p,p;Z) \tag{10}$$

satisfying

$$T^t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{11}$$

By using the following Morita transformation rules [4,10]:

$$\begin{aligned} G'_s &= \sqrt{detA}G_s, \qquad G' = AGA^t, \qquad A \equiv a + b\Theta, \\ F' + \Phi' &= A(F+\Phi)A^t \quad , \quad {F'}_{0i} = (FA^t)_{0i}, \end{aligned} \tag{12}$$

and [9]

$$STr'_{\theta'} = \frac{1}{\sqrt{\det A}} STr_\theta. \tag{13}$$

Ryang proved the invariance of $\mathcal{L}_{DBI}$. He also discussed the Morita transformation law of R-R potentials in a complicated way.

However, if we consider the interpolation also in the $\hat{\mathcal{L}}_{wz}$ term, we will get much simpler Morita transformation rules for R-R potentials. Concretly assume

$$\hat{\mathcal{L}}_{wz} = STr_\theta(e^{2\pi\alpha'(\hat{F}+\Phi)} \sum C_{(n)}), \tag{14}$$

then we have

$$\begin{aligned} C' &= \frac{1}{\sqrt{\det A}} C, \quad C'_i = \frac{1}{\sqrt{\det A}} A_i^a C_a, \quad C'_{ij} = \frac{1}{\sqrt{\det A}} (ACA^t)_{ij}, \\ C'_{ijk} &= \frac{1}{\sqrt{\det A}} (A^a_{[i} A^b_j A^c_{k]}) C_{abc}, \quad C'_{ijkl} = \frac{1}{\sqrt{\det A}} (A^a_{[i} A^b_j A^c_k A^d_{l]}) C_{abcd}, \\ {C'}_{ijklm} &= \frac{1}{\sqrt{\det A}} (A^a_{[i} A^b_j A^c_k A^d_l A^e_{m]}) C_{abcde}. \end{aligned} \tag{15}$$

Similar to Yang-Mills fields, when the θ varies, the R-R potentials should change as following to guarantee the correct interpolation:
In IIA case,

$$\begin{aligned}
\delta\hat{C}(\theta) &= -\hat{F}_{0i}^{-1}\delta\hat{F}_{0i}\hat{C}, \\
\delta\hat{C}_{jk}(\theta) &= -\delta(\hat{F}+\Phi)_{jk}\hat{C} - \hat{F}_{0i}^{-1}\delta\hat{F}_{0i}\hat{C}_{jk}, \\
\delta\hat{C}_{jklm}(\theta) &= -6\delta(\hat{F}+\Phi)_{jk}\hat{C}_{lm} - \hat{F}_{0i}^{-1}\delta\hat{F}_{0i}\hat{C}_{jklm},
\end{aligned} \tag{16}$$

and in IIB case,

$$\begin{aligned}
\delta\hat{C}_{j}(\theta) &= -\hat{F}_{0i}^{-1}\delta\hat{F}_{0i}\hat{C}_{j}, \\
\delta\hat{C}_{jkl}(\theta) &= -3\delta(\hat{F}+\Phi)_{jk}\hat{C}_{l} - \hat{F}_{0i}^{-1}\delta\hat{F}_{0i}\hat{C}_{jkl}, \\
\delta\hat{C}_{jklmn}(\theta) &= -10\delta(\hat{F}+\Phi)_{jk}\hat{C}_{lmn} - \hat{F}_{0i}^{-1}\delta\hat{F}_{0i}\hat{C}_{jklmn},
\end{aligned} \tag{17}$$

where $\delta\hat{F}_{ij}(\theta)$ and $\delta\Phi_{ij}(\theta)$ follow from reference [4]:

$$\begin{aligned}
\delta\hat{F}_{ij}(\theta) &= \frac{1}{4}\delta\theta^{kl}\{2\hat{F}_{ik}\star\hat{F}_{jl} + 2\hat{F}_{jl}\star\hat{F}_{ik} - \hat{A}_k\star(\hat{D}_l\hat{F}_{ij}+\partial_l\hat{F}_{ij}) - (\hat{D}_l\hat{F}_{ij}+\partial_l\hat{F}_{ij})\star\hat{A}_k\}, \\
\delta\Phi_{ij}(\theta) &= \delta\theta^{kl}(\hat{G}_{ik}\hat{G}_{lj}+\Phi_{ik}\Phi_{lj}).
\end{aligned} \tag{18}$$

We can also find that the Morita equivalence in the Hamiltonian becomes simpler when we consider the interpolation with modulus Φ in the whole action. From our Lagrangian (4) and (5) it is easy to obtain

$$\mathcal{H} = G_{00}^{1/2} STr_\theta\left\{\frac{-1}{G_s^2(2\pi)^{2p}\alpha'^{p+1}}det_{(p)}(G+\mathcal{F}) + \tilde{\epsilon}^t(G-\mathcal{F}G^{-1}\mathcal{F})\tilde{\epsilon}\right\}^{1/2}, \tag{19}$$

in which $\det_{(p)}$ stands for the $p\times p$ matrix(without time component) and

$$\tilde{\epsilon}^i = \epsilon^i - p^i_{(p)}(\hat{C},\mathcal{F}), \qquad p^i_{(p)}(\hat{C},\mathcal{F}) \equiv \frac{\partial P^i_{(p)}(\hat{C},\mathcal{F})}{2\pi\alpha'\partial\hat{F}_{0i}}, \tag{20}$$

where $\epsilon^i = \frac{\partial L}{\partial\hat{F}_{0i}}$ are canonical momenta. It can be checked that $\det_p(G+\mathcal{F})$ is equal to the trace term in reference [9] for $p=4$. Because of equation (12) and equation (13) the first part of Hamiltonian is obviouly invariant under Morita transformation. The same is true if $\tilde{\epsilon}^i$ transform as follows

$$\tilde{\epsilon'}^i = \sqrt{\det A}[(A^{-1})^t\tilde{\epsilon}]^i. \tag{21}$$

From the transformation of $\dot{A}_i = \hat{F}_{0i}$ in equation (12) and note the equation (13), we realize that

$$\epsilon'^i = \sqrt{\det A}[(A^{-1})^t\epsilon]^i. \tag{22}$$

To prove that

$$p'^i_{(p)} = \sqrt{\det A}[(A^{-1})^t p_{(p)}]^i. \tag{23}$$

we can choose $D=4$ as an example. From equation (14) we have

$$ {}^{\star}\hat{C}'^{i}_{(3)} \equiv \frac{1}{3!}\epsilon^{ijkl}\hat{C}'_{jkl} = \sqrt{\det A}[(A^{-1})^t\, {}^{\star}\hat{C}_{(3)})]^i, \tag{24} $$

and

$$ {}^{\star}(\hat{F}+\Phi)'^{ij} \equiv \frac{1}{2}\epsilon^{ijkl}(\hat{F}+\Phi)'_{kl} = (\det A)((A^{-1})^t\, {}^{\star}(\hat{F}+\Phi)A^{-1})^{ij}. \tag{25} $$

At last we get

$$ \begin{aligned} p'_{(4)} &= {}^{\star}\hat{C}'_{(3)} + {}^{\star}(\hat{F}+\Phi)'\hat{C}'_{(1)} = \sqrt{\det A}(A^{-1})^t({}^{\star}\hat{C}_{(3)} + (\hat{F}+\Phi)\hat{C}_{(1)}) \\ &= \sqrt{\det A}(A^{-1})^t\, p_{(4)}. \end{aligned} \tag{26} $$

Noncommutative parameter $\Theta(\theta)$ can be interpolated to zero, and $\Theta(\theta)$ can also be reduced to zero through Morita transformation. But the latter case (different from the former) is restricted to the rational value. Interpolating parameter must be very small, while the Morita parameter may not have this restriction. It was argued that modulus Φ appears as a magnetic backgroud. Under this magnetic background the Morita transformation becomes simpler. That is due to the Morita symmetry of Φ compensating the inhomogeneous change of gauge field. More property of Φ can be found from Reference [6,11].

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